

ENCRYPTION WITH WEAKLY RANDOM KEYS USING A QUANTUM CIPHERTEXT

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The lack of perfect randomness can cause significant problems in securing communication between two parties. McInnes and Pinkas [13] proved that unconditionally secure encryption is impossible when the key is sampled from a weak random source. The adversary can always gain some information about the plaintext, regardless of the cryptosystem design. Most notably, the adversary can obtain full information about the plaintext if he has access to just two bits of information about the source (irrespective on length of the key). In this paper we show that for every weak random source there is a cryptosystem with a classical plaintext, a classical key, and a quantum ciphertext that bounds the adversary's probability p to guess correctly the plaintext strictly under the McInnes-Pinkas bound, except for a single case, where it coincides with the bound. In addition, regardless of the source of randomness, the adversary's probability p is strictly smaller than 1 as long as there is some uncertainty in the key (Shannon/min-entropy is non-zero). These results are another demonstration that quantum information processing can solve cryptographic tasks with strictly higher security than classical information processing.

1 Introduction

Random numbers play a crucial role in many areas of computer science, e.g. randomized algorithms and cryptography. Real world random number generators deliver imperfect (biased) randomness and a number of theoretical models of imperfect random number sources [2, 7, 15, 19] were introduced to study possibilities to obtain perfect randomness through their software postprocessing [16]. Importance of this post-processing stems from the fact that many of its applications were designed to use perfect randomness, and, in fact, vitally require it for a reasonable performance. Devices based on quantum mechanical properties should theoretically serve as sources of ideal randomness [3, 8] or even as unconditionally secure cryptosystems [1, 9, 12, 17, 18]. However, in real conditions they are extremely sensitive to the influence of environment and rely on classical post-processing of the measurement results. Even after these procedures the outcome is far from being perfect (see e.g. [11] and references therein).

Cryptography counts among fields that are highly sensitive to the quality of randomness used (see. e.g. [4]). One of the most prominent results showing the influence of weakness of randomness is by

McInnes and Pinkas [13] proving that there is no perfectly secure encryption scheme when the key is sampled from a weakly random source (e.g. min-entropy [7]). The authors also derived a tight (minimal and achievable) bound on probability that the adversary can determine the plaintext, when an arbitrary encryption system is used, but the key is sampled from a min-entropy source. This bound is a function of $c = l - b$, where l is the key length and b is its min-entropy (see Section 2 for definition of min-entropy).

In this paper we propose an encryption of classical information using classical key and quantum channel (i.e. ciphertext is a quantum state) such that the adversary's probability to determine the plaintext is strictly smaller than the McInnes-Pinkas bound for all values $c = l - b \geq 0$, except for values $c = 0$ and $c = 1$, where it coincides with the bound.

The paper is organized as follows. In the second section we introduce basic definitions and recall the result by McInnes and Pinkas in detail. In the third section we introduce the encryption onto a quantum ciphertext. In the fourth section we derive the maximal security for the approximation of a continuous key. In the fifth section we deal with the consequences of the discretisation of the random key, whereas in the last section we summarize our results and conclude.

2 Preliminaries

The min-entropy of the probability distribution (random variable, source) \mathbf{Z} is defined by

$$H_\infty(\mathbf{Z}) = \min_{z \in \mathcal{Z}} (-\log \Pr(\mathbf{Z} = z)). \quad (1)$$

We denote a source as (l, b) -source if it is emitting l -bit strings drawn according to a probability distribution with min-entropy at least b . Thus, every specific l -bit sequence is drawn with probability smaller or equal to 2^{-b} . Notice that for $b = l - c$, the probability of each l -bit string is upper bounded by $2^c \frac{1}{2^l}$, and parameter c is called min-entropy loss.

A source is (l, b) -flat iff it is an (l, b) source and it is uniform on some subset of 2^b sample points i.e., all probabilities are either 0 or 2^{-b} .

We are going to consider the following scenario. Alice and Bob share a secret key k that is used to determine the encoding (decoding) function. In this setting the plaintext is a single, uniformly distributed bit and the ciphertext is (arbitrarily long, but finite) bitstring. The encryption system is specified by the set of encoding rules

$$e_k : X \rightarrow Y \quad (2)$$

parameterized by the keys $k \in K$, with $X = \{0, 1\}$ being the set of plaintexts and Y being the set of ciphertexts. To each encoding function there is the corresponding decoding function d_k that perfectly recovers the original input, i.e.

$$\forall k \in K \ d_k \circ e_k = id_X. \quad (3)$$

Let \mathbf{X} be the random variable describing the probability distribution of the plaintext and \mathbf{X}' random variable describing adversary's estimate given by his decoding strategy. Security is parameterized by adversary's ability to recover the original plaintext

$$p = \sum_{x \in X} \Pr(\mathbf{X}' = x | \mathbf{X} = x) \Pr(\mathbf{X} = x). \quad (4)$$

In the McInnes and Pinkas paper, for a given length of the key l and a parameter c the key is distributed according to an $(l, l - c)$ distribution. Parameters l and c are part of the cryptosystem design, the adversary is assumed to know the actual (biased) distribution \mathbf{K} of the key (but not the value k of the key). The probability p to reveal the plaintext the adversary can achieve for an arbitrary cryptosystem, a suitable $(l, l - c)$ source and a uniformly distributed plaintext is lower bounded by

$$p \geq \begin{cases} 1 & \text{for } 2 \leq c \leq l \\ \frac{1}{2} + \frac{2^c}{8} & \text{for } 2 - \log_2 3 \leq c \leq 2 \\ \frac{2^c}{2} & \text{for } 0 \leq c \leq 2 - \log_2 3. \end{cases} \quad (5)$$

In particular, they have shown that the (maximal achievable) security^a is independent of l and no security can be achieved if $c \geq 2$.

3 Quantum ciphertext

In our solution we consider encoding classical plaintext to a quantum ciphertext. The encoding function is of the form

$$e_k : X \rightarrow S(\mathcal{H}) \quad (6)$$

parameterized by the key $k \in K$, with $X = \{0, 1\}$ being the set of plaintexts, \mathcal{H} a suitable Hilbert space and $S(\mathcal{H})$ the set of (possibly mixed) states on the Hilbert space \mathcal{H} , i.e. positive trace one operators acting on \mathcal{H} . In our further analysis we limit ourselves to only a single quantum bit, i.e. to two-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_2$. Extensions to higher dimensional Hilbert spaces will be discussed in the conclusion.

The decoding procedure consists of measurement of the received quantum system aiming to distinguish between states $e_k(0)$ and $e_k(1)$. The correctness requirement gives (regardless of \mathcal{H}) that for every k the states $e_k(0)$ and $e_k(1)$ must be orthogonal. For a qubit, this is only possible if all of them are pure states. Let us use the notation $e_k(0) = |\psi_k\rangle$ and $e_k(1) = |\phi_k\rangle$, with the orthogonality condition

$$\langle \phi_k | \psi_k \rangle = 0 \quad (7)$$

for all k 's.

To obtain the encoded bit, Bob (knowing the key) adjusts his measurement device accordingly to obtain a well defined result discriminating between $|\psi_k\rangle$ and $|\phi_k\rangle$. For this purpose a standard von Neumann measurement is fully sufficient. On the other hand, the adversary's knowledge is limited to the (known) probability distribution on keys. He has to discriminate between the average states

$$\rho_0 = \sum_{k \in K} P(\mathbf{K} = k) |\psi_k\rangle \langle \psi_k| \text{ and } \rho_1 = \sum_{k \in K} P(\mathbf{K} = k) |\phi_k\rangle \langle \phi_k|. \quad (8)$$

Analogously to the classical case, the effectiveness of the adversary's strategy is given by the probability p given by Eq. (4). The adversary's strategy is thus to maximize this probability by performing a minimum error measurement, which is a simple two outcome von Neumann measurement [5, 6, 14].

Without the loss of generality we can choose a basis so that the state ρ_0 (8) has the form

$$\rho_0 = \begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix} \quad (9)$$

with $a \geq \frac{1}{2}$. Due to the orthogonality condition (7) the state ρ_1 has the form

$$\rho_1 = \begin{pmatrix} 1-a & 0 \\ 0 & a \end{pmatrix}. \quad (10)$$

The optimal measurement of the attacker is thus just the spin z projection measurement and the probability to get a correct outcome (determine the original plaintext) then reads

$$p = \frac{1}{2} \left[\frac{1}{2} \text{Tr} |\rho_0 - \rho_1| + 1 \right] = a. \quad (11)$$

^aProbability of making a correct guess by the adversary is bounded from below by the formula (5) independently of l . However, such security is achievable only for a cryptosystem with high enough l , for smaller l the achievable probability for the attacker rises further.

4 Continuous code

In this section we assume that Alice and Bob share a random key that is continuous, e.g. a single complex number. Such a coding can uniformly cover the state space of a single qubit \mathcal{H} , which can be depicted as a Bloch sphere. Later, in Section 5, we will show that there exists a discrete coding that approximates the continuous coding introduced in this section with an arbitrary precision (with respect to adversary's probability to reveal the plaintext).

Suppose that Alice and Bob know (or expect) that the random key they share can be biased by a certain amount. Their aim is to choose the coding in such a way that any partial knowledge about the key by an eavesdropper would lead to as small probability of obtaining the correct encrypted bit as possible. This is naturally achieved by a smooth coverage of the state space (Bloch sphere) in such a way that the probability density of selected states will be equal on all points of the sphere.

The first important observation is that we can fix an arbitrary adversary's measurement $P = |0\rangle\langle 0|$ by fixing the basis and determine the key distribution that is optimally distinguished by the measurement. This can be done as all measurements are unitarily equivalent, i.e. for each pair of measurements there is a unitary rotation of the sphere that maps one measurement to the other. Hence, if there is an optimal distribution for one measurement, for any other measurement there exists a (different) distribution giving the same result.

Let us define a source with min-entropy loss at most c for continuous spaces. According to (1) a discrete source with length l and min-entropy $H_\infty(\mathbf{Z})$ has min-entropy loss $c = l - H_\infty(\mathbf{Z})$:

$$\begin{aligned} c &= l - H_\infty(\mathbf{Z}) \\ &= l - \min_{z \in \mathbf{Z}} (-\log \Pr(\mathbf{Z} = z)) \\ &= \max_{z \in \mathbf{Z}} (l + \log \Pr(\mathbf{Z} = z)) \\ &= \max_{z \in \mathbf{Z}} (\log(|\mathbf{Z}| \Pr(\mathbf{Z} = z))). \end{aligned} \quad (12)$$

Equation (12) can be easily extended to continuous space by changing maximalization to sup. $|\mathbf{Z}|$ has to be changed to the volume of the probability space \mathcal{H} and probability function \Pr becomes probability density function μ .

Now let us define a continuous weak source over space \mathcal{H} with min entropy loss c as a set of probability density functions for which

$$c = \sup_{\psi \in \mathcal{H}} \log(|\mathcal{H}| \mu(\psi)). \quad (13)$$

After additional simplification it is easy to see that the condition reads $\sup_{\psi \in \mathcal{H}} \mu(\psi) = \frac{2^c}{|\mathcal{H}|}$.

Let us consider all distributions on the sphere such that for any state on the sphere its probability density is at most $2^c |\mathcal{H}|^{-1}$, with $|\mathcal{H}|$ being the area of the surface of the Bloch sphere. Later, in Section 5, we show that such distributions are analogous to discrete min-entropy sources. Flat distributions correspond to continuous distributions where only a 2^{-c} fraction of all possible keys would appear with equal nonzero probability density.

Let us now examine a situation, where the adversary prepares for Alice and Bob a flat distribution on the subset of size $2^{-c} |\mathcal{H}|$ (i.e. the keys are selected with equal probabilities from 2^{-c} fraction of all possible keys). We propose a distribution $\mu_{opt}(|\phi\rangle)$ defined as

$$\mu_{opt}(|\phi\rangle) = \begin{cases} \frac{1}{2^{-c} |\mathcal{H}|} & \text{for } |\phi\rangle \in Y \\ 0 & \text{elsewhere} \end{cases}$$

for $Y = \{|\phi\rangle \in \mathcal{H}; |\langle 0|\phi\rangle|^2 \geq g; |Y| = 2^{-c} |\mathcal{H}|\}$ for some suitable constant g dependent on c . Let us assume that Alice encodes 0 (uniformly at random) into one of the states from Y . The probability to obtain the correct outcome of the measurement P is

$$p_{opt} = \int_{\mathcal{H}} \mu_{opt}(|\phi\rangle) |\langle 0|\phi\rangle|^2 d\phi = \frac{1}{2^{-c} |\mathcal{H}|} \int_{|\phi\rangle \in Y} |\langle 0|\phi\rangle|^2 d\phi. \quad (14)$$

Let $\mu(|\phi\rangle)$ be any distribution with $\mu(|\phi\rangle) \leq 2^c |\mathcal{H}|^{-1}$ everywhere, i.e. we restrict ourselves to distributions corresponding to discrete min-entropy $(l, l - c)$ distributions. We will now prove that the distribution μ_{opt} is optimal among these distributions, i.e there is no other distribution such that the adversary can obtain a higher probability to detect the correct ciphertext. Let us calculate this probability for a general distribution $\mu(|\phi\rangle)$

$$\begin{aligned}
p &= \int_{\mathcal{H}} \mu(|\phi\rangle) |\langle 0|\phi\rangle|^2 d\phi \\
&= \int_{|\phi\rangle \in Y} \mu(|\phi\rangle) |\langle 0|\phi\rangle|^2 d\phi + \int_{|\phi\rangle \in \setminus Y} \mu(|\phi\rangle) |\langle 0|\phi\rangle|^2 d\phi \\
&= \int_{|\phi\rangle \in Y} \mu_{opt}(|\phi\rangle) |\langle 0|\phi\rangle|^2 d\phi + \int_{|\phi\rangle \in Y} [\mu(|\phi\rangle) - \mu_{opt}(|\phi\rangle)] |\langle 0|\phi\rangle|^2 d\phi + \int_{|\phi\rangle \in \setminus Y} \mu(|\phi\rangle) |\langle 0|\phi\rangle|^2 d\phi \\
&= p_{opt} + \int_{|\phi\rangle \in Y} [\mu(|\phi\rangle) - \mu_{opt}(|\phi\rangle)] |\langle 0|\phi\rangle|^2 d\phi + \int_{|\phi\rangle \in \setminus Y} \mu(|\phi\rangle) |\langle 0|\phi\rangle|^2 d\phi, \tag{15}
\end{aligned}$$

where $\setminus Y = \mathcal{H} - Y$.

Let us now analyze the two remaining terms in the Eq. (15). As $\mu(|\phi\rangle) \leq \mu_{opt}(|\phi\rangle)$ for all $|\phi\rangle \in Y$ (recall that $\mu_{opt}(|\phi\rangle)$ reaches the maximal permitted value on the subset Y), the integrated function is non-positive on the whole area of integration (Y), $|\langle 0|\phi\rangle|^2 \geq g$ on the whole area as well and thus the first integral can be bounded from above by $g \int_{|\phi\rangle \in Y} [\mu(|\phi\rangle) - \mu_{opt}(|\phi\rangle)] d\phi$. In the last term the integrated function is positive, but $|\langle 0|\phi\rangle|^2 < g$ on the whole area and thus the second integral can be also bounded from above by $g \int_{|\phi\rangle \in \setminus Y} \mu(|\phi\rangle) d\phi$. Altogether we get

$$p \leq p_{opt} + g \int_{|\phi\rangle \in Y} [\mu(|\phi\rangle) - \mu_{opt}(|\phi\rangle)] d\phi + g \int_{|\phi\rangle \in \setminus Y} \mu(|\phi\rangle) d\phi.$$

As $\mu_{opt}(|\phi\rangle)$ is 0 outside Y , we can include it into the integration outside Y with a minus sign. Now we join the integration through the whole space and using the normalization condition on both distributions we get

$$p \leq p_{opt} + g \int_{\mathcal{H}} [\mu(|\phi\rangle) - \mu_{opt}(|\phi\rangle)] d\phi = p_{opt}.$$

This completes the proof of the optimality of the flat $\mu_{opt}(|\phi\rangle)$ distribution.

Recall that the state space of a single qubit represented as a Bloch sphere is proportional to a unit sphere with the surface equal to $|\mathcal{H}| = 4\pi$. The set Y derived above is a spherical cap on the Bloch sphere with the center in $|0\rangle$. The desired flat distribution is hence equivalent to the uniform distribution on a spherical cap with a surface $4\pi 2^{-c}$. The height of such a spherical cap is $h = 2^{-c+1}$ (reaching from 2 for $c = 0$ to 0 for large c). The average state observed by the attacker in the case of the plaintext 0 is the center of the mass of the surface of the spherical cap, which is on the axis of the cap at the height $h/2$. The average state then reads

$$\rho_0 = \frac{1}{2} \mathbb{I} + \frac{h}{2} \sigma_z = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| \tag{16}$$

with

$$p = 1 - h/4 = 1 - 2^{-c-1}. \tag{17}$$

The appearance of $h/4$ instead of $h/2$ is due to the renormalization of the axis: while the Bloch sphere has diameter 2, the parameter p changes from 0 to 1 across the sphere.

Accordingly, the state observed by the attacker in the case of sending 1 is $\rho_1 = (1-p)|0\rangle\langle 0| + p|1\rangle\langle 1|$. Observing that $p \geq 1/2$ and substituting into Eq. (11), the optimal adversary's probability for this state is also p .

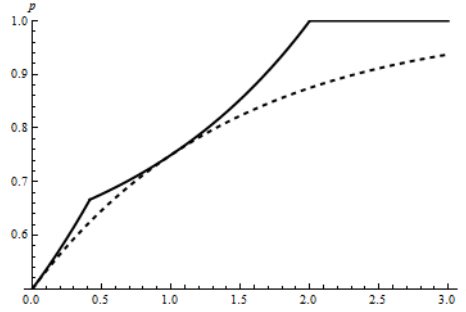


Fig. 1. Comparison of the probability of a successful guess by the attacker, if Alice and Bob use classical channel (full line) and quantum channel (dashed line).

The comparison of classical bounds and quantum approach is given in the Fig. (1). It is clear that the probability of the adversary to correctly guess the ciphertext is for all parameters c (except 0 and 1) strictly better than classical. Even more interesting, the probability is non-vanishing even for large c , what means that (some) privacy is established even in the case when the adversary has almost perfect control of the "random" key.

5 Discrete code

In this section we will show that for any $c \geq 0$ and an arbitrarily small $\epsilon > 0$ there exists an encryption system

$$e_k : \{0, 1\} \rightarrow S(\mathcal{H}) \quad (18)$$

indexed by keys from a finite set K that bounds the adversary's probability to determine the original plaintext by $1 - 2^{-c-1} + \epsilon$.

All the derivations made in the previous section were done for continuous distribution of the states $|\psi\rangle$ over the state space. If we consider a discrete finite number n of possible keys $|\psi_i\rangle$, we can only approximate such distributions using a finite number of lattice points [10] uniformly distributed over the surface of the Bloch sphere.

Let us assign to each single state $|\psi_i\rangle$ its neighborhood, i.e. a subset \mathcal{H}_i of the whole Hilbert space such that

- (i) The surface on the Bloch sphere of each \mathcal{H}_i equals to $\frac{4\pi}{n}$.
- (ii) Neighborhoods corresponding to different states are disjoint, i.e. $\forall i, j \ |\psi_i\rangle \neq |\psi_j\rangle \Rightarrow \mathcal{H}_i \cap \mathcal{H}_j = \emptyset$.
- (iii) The distance between a state $|\psi_i\rangle$ and any state in its neighborhood measured in the relative angle on the Bloch sphere is upper bounded by $O(n^{-1/2})$.

For every n , there exist a set of states $|\psi_i\rangle$ and its neighborhoods that fulfill the aforementioned conditions. This can be seen from the fact that even a suitable (by far non-optimal) distribution of points on \sqrt{n} meridians with \sqrt{n} points each yields a maximal angle of $\frac{3\pi}{2\sqrt{n}}$ (for explanation see Fig. (2)), which fulfills the third condition, whereas the first two can be fulfilled trivially.

We will show that for any probability distribution on the n aforementioned states with min-entropy at least $\log_2(n) - c$, the adversary's probability to determine the plaintext is at most $p_{opt} + \epsilon$, where ϵ drops as $n^{-1/2}$. Let us fix a particular distribution $(q_i)_{i=1}^n$ on states $|\psi_i\rangle$. We construct a continuous distribution $\mu_q(|\phi\rangle) = \frac{nq_i}{4\pi}$ for $\phi \in \mathcal{H}_i$. The adversary's probability to obtain the plaintext in the case of distribution $(q_i)_{i=1}^n$ on states $|\psi_i\rangle$ reads (compare to Eq. (14))

$$q_n = \sum_{i=1}^n q_i |\langle 0 | \psi_i \rangle|^2. \quad (19)$$

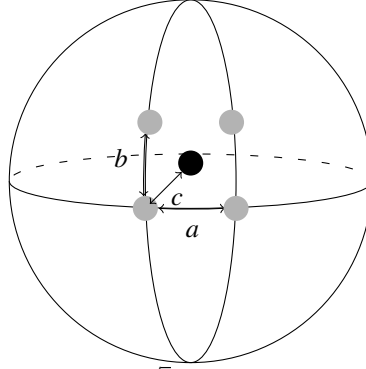


Fig. 2. As there are \sqrt{n} meridians used, there are \sqrt{n} points on the equator. Naturally the most distant points are in the vicinity of the equator, as there are the meridians most distant from each other. On the picture, four points are depicted, two of them on meridian and two just one position above. The most distant point to any of these points is in the center of the formation formed by these four points. The distance c is bounded from above by the triangle inequality by its distance to the equator $\frac{b}{2} = \frac{\pi}{2\sqrt{n}}$ and half of the distance between two points on the equator being $\frac{a}{2} = \frac{\pi}{\sqrt{n}}$, thus together $\frac{3\pi}{2\sqrt{n}}$.

As the difference of the fidelity of the state $|\psi_i\rangle$ with $|0\rangle$ and any state from its neighborhood with $|0\rangle$ is upper bounded by $O(n^{-1/2})$, we can approximate Eq. (19) by a correctly normalized integral over its neighborhood

$$|\langle 0|\psi_i\rangle|^2 \leq \frac{n}{4\pi} \int_{|\phi\rangle \in \mathcal{H}_i} [|\langle 0|\phi\rangle|^2 + O(n^{-1/2})] d\phi. \quad (20)$$

Substituting we get

$$\begin{aligned} q_n &\leq \sum_{i=1}^n q_i \frac{n}{4\pi} \int_{|\phi\rangle \in \mathcal{H}_i} [|\langle 0|\phi\rangle|^2 + O(n^{-1/2})] d\phi \\ &= \sum_{i=1}^n \int_{|\phi\rangle \in \mathcal{H}_i} \left[q_i \frac{n}{4\pi} |\langle 0|\phi\rangle|^2 + q_i \frac{n}{4\pi} O(n^{-1/2}) \right] d\phi \\ &= \int_{|\phi\rangle \in \mathcal{H}} \mu_q(|\phi\rangle) |\langle 0|\phi\rangle|^2 d\phi + \sum_{i=1}^n \left(q_i \frac{n}{4\pi} O(n^{-1/2}) \int_{|\phi\rangle \in \mathcal{H}_i} d\phi \right) \\ &\leq p_{opt} + \sum_{i=1}^n (q_i O(n^{-1/2})) \end{aligned} \quad (21)$$

$$\begin{aligned} &= p_{opt} + O(n^{-1/2}) \\ &\leq [1 + O(n^{-1/2})] p_{opt}. \end{aligned} \quad (22)$$

Inequality (21) holds, because μ_q satisfies all min-entropy requirements set in the previous section, and thus the integral can be bounded from above by p_{opt} . The last inequality holds, because p_{opt} is a positive constant, not depending on n .

We can conclude that for any $c \geq 0$ and an arbitrarily small $\epsilon > 0$ there exist a (sufficiently large) n that bounds the adversary's probability to determine the original plaintext by $1 - 2^{-c-1} + \epsilon$. It is obvious that for any fixed c a suitable ϵ can be chosen such that the probability of the adversary to guess the ciphertext correctly is strictly smaller than one.

Moreover, consider a cryptosystem with only two elements $k_1, k_2 \in K$ such that $Pr(\mathbf{K} = k_1) > 0$,

$Pr(\mathbf{K} = k_2) > 0$. Then the average states ρ_0, ρ_1 are not pure, i.e. (in a suitable basis)

$$\rho_0 = \begin{pmatrix} a & 0 \\ 0 & 1-a \end{pmatrix} \quad (23)$$

with $a, 1-a > 0$. This implies the quantity in Eq. (11) is strictly smaller than 1. Thus, for any encoding the adversary's probability to determine the original plaintext is strictly smaller than 1, as long as there are at least two keys with nonzero probability.^b

6 Conclusion

In this paper we have shown that a quantum ciphertext can significantly increase the security of classical communication between two parties using weak random sources. In particular, for significantly weak sources, where less than a quarter of keys can be used for encryption, no security can be achieved with a classical ciphertext. This is true independently both on the length of the key l and the length of the ciphertext. On the contrary, for quantum ciphertexts, some level of security can be achieved even if only two keys appear with non-zero probability.

For all sources (for all values of c) the presented quantum approach is at least as good as the classical one. Moreover, except for $c = 0$ (trivial case with perfect sources and perfect security) and $c = 1$ the quantum approach outperforms the classical for all values of c .

It is important to stress that our result does not give a lower bound on the security one can achieve in this problem in general. Using encoding into systems of higher dimension (e.g. a pair of qubits or a qutrit) may achieve significantly better results, as more complicated encodings also using mixed states come into question.

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^bEquivalently: Shannon (or min-entropy) of the key is non-zero.

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